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SOME LIFT AND STRUCTUR FROM A MANIFOLD TO ITS TANGENT BUNDLE

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Abstract:

In this paper, Lift of Tensor field, almost complex structures, F-structure and Para compact structures are studied.

1. Lift of tensor fields:

Let V be a n-dimensional differentiable manifold and $T(V)$ is its tangent bundle .The projection being denoted by $\pi : T(V) \to V$ for a differentiable function ψ in V. The function $\pi^* \psi$ induced from ψ in $T(V)$ is denoted by

(1.1) $$\pi^V \psi = \pi^* \psi$$

and is called the vertical lift of function ψ . Any 1-form ω given in differentiable manifold V is, in natural way, regarded as a function in $T(V)$, which we be denoted by $\iota\omega$. If we are given a vector field X in V, then we define a vector field X^V in $T(V)$ by

(1.2) $$X^V(\iota\omega) = (\omega(X))^V$$

ω being an arbitrary $1-form$ in V. The vector Field X^V thus defined is called the vertical lift of the vector field X.

We define the vertical lift of (1.1) type $d\psi$ and $\theta d\psi$ by (1.3) by

(1.3) $(d\psi)^V = d(\psi)^V$ and $(\theta d\psi)^V = \theta^V (d\psi)^V$ respectively ψ and θ being arbitrary function in V and the vertical lift ω^V of an arbitrary $1-form\ \omega$ in V by

(1.4) $\omega^V = (\omega_\alpha)^V (dx^\alpha)^V$

In each open set $\pi^{-1}(U)$, where U is a co-ordinates neighborhood with local coordinates (X^h) in V and ω is given by $\omega = \omega_\alpha dx^\alpha$

in U. It is easily verified that the vertical lift ω^V of $1-form$ defied by (1.4) in each $\pi^{-1}(U)$ is a global $1-form$ in tangent bundle. When there is given function ψ in V, we put

(1.5) $\psi^H = \psi^C - \nabla_\gamma \psi$ In $T(V)$, where $\nabla_\gamma \psi = (\nabla_\gamma \psi)$ and all the functions ψ^H defined in $T(V)$. The horizontal lift of the function ψ for a vector field X given in V. we define a vector field in tangent bundle by

(1.6) $X^H \psi^H = X^C \psi^H - (\nabla_\gamma X)\psi^H$ ψ being an arbitrary function in V and call ψ^H the horizontal lift of vector field X, where ∇ denotes the affine connection.

Given $1-form$ ψ in V , we define ω^H in $T(V)$ by

(1.7) $\omega^H (X^H) = 0$ X being an arbitrary vector field in V, the $1-form$ ω^H thus defined in $T(V)$ is called the Horizontal lift

(1.8) $F^H X^H = (FX)^H$ and $F^X X^H = 0$ for any $X \in V$ We define Vertical lift S^V and Horizontal lift S^H in $T(V)$ for a tensor field S of type (1.2) given in V as

(1.9) $S^V (X^C, Y^C) = (S(X,Y))^V$ and $S^H (X^H, Y^H) = (S(X,Y))^H$

X and Y being arbitrary vector fields in V.We shall now give local representation of the lifts. Let $(U(X^h))$ be a coordinate neighborhood of differential manifold V , where (x^h) is a system of local coordinates defined in U. Let (y^h) be the system of Cartesian coordinates in each tangent space $T_P(V)$ of V at P with respect to $\frac{\partial}{\partial x^h}$, where P is an arbitrary point in U, there is an open set $\pi^{-1}(U)$ of $T(U)$,where $p \in U$,then in the open set $\pi^{-1}(U)$ of $T(U)$, we can introduce local coordinates (x^h, y^h), which are called coordinate induced in $\pi^{-1}(U)$ from (x^h). Let there be a given function $\psi(x^h)$ in U, then its vertical lift ψ^V and its horizontal lift ψ^H represented as

(1.10) $\psi^V = \psi(x^h)$ and $\psi^H = \psi^C - \nabla_\gamma \psi(x^h)$ In $\pi^{-1}(U)$ with respect to induced coordinates (x^h, y^h) where $\partial = y^\alpha \dfrac{\partial}{\partial x^\alpha}$

If a vector field X has components X^H in U, then its vertical lift X^V and its horizontal lift X^H have respectively component of the form (1.11) $X^V : \begin{pmatrix} 0 \\ x^h \end{pmatrix}$ and $X^H : \begin{pmatrix} X^h \\ -\Gamma^h_\alpha x^\alpha \end{pmatrix}$

In $\pi^{-1}(U)$ with respect to induced coordinates (x^h, y^h) If $1-$ *form* v has components $\omega_\alpha \in U$, hen its vertical lifts ω^V and its horizontal lifts ω^H have respectively components of the form

(1.12) $\omega^V : (\omega_\alpha, 0)$ and $\omega^H : (\Gamma^h_\alpha \omega_\alpha, \omega_\alpha)$ in $\pi^{-1}(U)$ with respect to induced coordinates (x^h, y^h).If a tensor field F of type (1,1)has components $F^h_\alpha \in U$, then its vertical lift F^V and its horizontal lift F^H have respectively components of the form

(1.13) $F^V : \begin{bmatrix} 0 & 0 \\ F^h_\alpha & 0 \end{bmatrix}$ and $F^H : \begin{bmatrix} F^H & 0 \\ -\Gamma^h_\beta F^\beta_\alpha + \Gamma^\beta_\alpha F^h_\beta & \Gamma^h_\alpha \end{bmatrix}$

in $\pi^{-1}(U)$ with respect to the induced coordinates (x^h, y^h). Thus the horizontal lift of the identity tensor field of V is the identity tensor field in $T(V)$. On taking account of the definition of lift or their local representation (1.10),(1.11),(.2) and(1.13), we get formulae

(1.14) $(\theta\psi)^V = \theta^V \psi^V, (\theta\psi)^H = \theta^H \psi^H$ $(\psi X)^V = \psi^V X^V$; $(\psi X)^H = \psi^H X$
$(\psi\omega)^V = \psi^V \omega^V$; $(\psi\omega)^H = \psi^H \omega^H$; $(\psi \otimes \omega)^H = \psi^H \otimes \omega^H$

For any function ψ, θ any vector field X form ω given in V

(1.15) $X^V \psi^V = 0$; $X^H \psi^H = (X\psi)^V$

$$X^H \psi^V = (X\psi)^C - \gamma((d\psi)(X)$$

$$X^H \psi^H = 0$$

Fr any function ψ and any vector field X given in V;

(1.16) $$\omega^V(X^V) = 0\,;\,\psi^V(X^H) = (\omega(X))^V$$

$$\omega^H(X^V) = (\omega(X))^V\,;\,\omega^H(X^H) = 0$$

For any vector field X and any 1-form ω given in V,

$$\left[X^V, Y^H\right] = \left[X, Y\right]^V - (\nabla_X Y)^V = -(\hat{\nabla}_Y X)^V$$

(1.17) $$\left[X^C, Y^H\right] - \gamma(\iota_X Y)$$

$$[X^H, Y^H] = [X, Y]^H - \hat{R}(X, Y)$$

For any vector field X and Y given in V, where ∇ denotes the affine connections in V defined by

$$\hat{\nabla} Y = \nabla_\gamma X + \left[X, Y\right]$$

For any two vector field X and Y given in V

(1.18) $$F^V X^V = 0; F^H X^V = (FX)^V$$

$$F^V(X^H) = (FX)^H\,,\,F^H X^H = (FX)^H$$

For any vector field X and any tensor field F of type (1,1) given in V. Le there be given any two tensor field F and K of type (1,1) in V , then we get

(1.19) $$F^H(K^H) = (FK)^H$$

Thus, if there are given a tensor field F of type (1, 1) and a polynomial $\phi(t)$; of a variable t, then we have from (1.19)

(1.20) $$\phi(F))^H = \phi(F)^H$$

For example

(1.21) $$(F^2 + I)^H = (F^H)^2 + I$$

$$(F^2 + F)^H = (F^H)^3 + F^H$$

Where I denotes the identity tensor field of type (1, 1) in the corresponding manifold V of T (V). Let there be a given a tensor field of type (1, 1) in V, then Nijenhuis tensor N is defined by

(1.22) $$N(X,Y)=[FX,FY]+F^2[X,Y]-F[FX,Y]-Fx,FY]$$

X and Y being arbitrary vector field in V.

On taking horizontal lift of both sides and taking account of1.9),(1.170 and (1.18), we have

(1.23) $$\mathrm{N}^H(X^H,Y^H)=[F^HX^H,F^HY^H]+(F^H)^2[X^H,Y^H]-F^H[F^HX^H,Y^H]$$

The right hand side of (1,23) is nothing but the Nijenhuis tensor $\tilde{N}(X^H,Y^H)$ of the horizontal lift F^H of F, thus we have

(1.24) $$\mathrm{N}^H=\tilde{N}$$

Where $\tilde{N}$ denotes the Nijenhuis tensor of F^H.

Let F be the tensor field of type (1,1) in V, then the vertical lift F^V is of the rank r and its horizontal lift F^H is of rank 2r iff F is of rank r.

Let there be a given projection tensor $\mathrm{m}\in\mathrm{V}$ such that $m^2=m$, then there exist a distribution D in V ,which is determined by m. On taking account (1.21) we get,

$$(\mathrm{m}^H)^2=m^H$$

which means m^H is also projection tensor in T(V), thus there exist in T(V)a distribution D^H corresponding to m^H and which we call the horizontal lift of distribution D. The horizontal lift D^H

by all vector field of type X^V and X^H, X being an arbitrary vector field belonging to the distribution D because we have

(1.25) $$\mathrm{m}^H X^V=(\mathrm{m}X)^V,\ \mathrm{m}^H X^H=(\mathrm{m}X)^H$$

The distribution D is integrated iff

(1.26) $$I(mX,mY)=0$$

For any vector field X and Y in V, where $I=I-m$, taking the horizontal lift of the both sides in(1.26), we obtain

$$I^H(m^H X^H, m^H Y^H) = 0$$

Theorem 1.1: Let there be a given distribution D with projection tenor in V, the horizontal lift m^H is integrable in T(V) if and only if D is integrable in V.

Proof: Let there be a given a differentiable transformation

$$\rho : V \to V$$

and denoted simply $\rho : T(V) \to T(V)$.

The differential of transformation $\rho : V \to V$ for a 1-form of $\omega \in V$, we define 1-form $\rho\omega$

$$(\rho\omega)X = \omega(\rho(X))$$

X being an arbitrary vector field in V, if the differential of the transformation $\rho : T(V) \to T(V)$ is denoted by $\rho^H : T(T(V)) \to T(V))$

Then we have the following formulae

$$(\rho X)^V = \rho^C X^V, (\rho X)^H = \rho^H X^H$$

$$(\rho\omega)^V = \rho^C \omega^V; (\rho\omega)^H = \rho^H \omega^H$$

For any vector field X and 1-form ω given in V for a tensor field F of type (1, 1) given in V, we define a tensor field ρF by

$$(\rho F)^X = (F(\rho X)$$

then we get

$$(\rho F)^V = \rho^C F^V, (\rho F)^H = \rho^H F^H$$

The tensor field ρF is that induced from the given F by the transformation $\rho : V \to V$.

2. Almost Complex Structure:

Let there be a given field F of type (1, 1) in V. On taking account of (1.21) we have

$$(F^H)^2 + I = 0$$

if and only of $(F^2) = -\mathrm{I}$, thus we have

Theorem2: The horizontal lift F^H of a tensor field of type F of type (1, 1) given in differentiable manifold V is an almost complex structure in T (V) if and only if so F in V.

Proof: A tensor field of type (1, 1) is called f—structure of rank r, when F satisfies $F^3 + F = 0$ or $F^3 = -F$ and the rank of is a constant r everywhere, r being necessarily even. On taking account of (1.24), we have the theorem,

Theorem 2.2: An almost complex structure F in an almost complex space V, a transformation $\rho : V \to V$ preserves the structure F if and only if its differential function $\rho : T(V) \to T(V)$ preserves F^H.

3. F-Structure:

Let there be given n dimensional differentiable manifold V of class C^∞ on V and a tensor of type F of type (1,1) of rank r $(1 \le r \le n)$ such that

(3.1) $F^3 + F = 0$, then F is called F-structure. If we put

(3.2) $I = -F^2, m^2 = F^2 + I$, then we have

$$I + m = I, I^2 = -IoF \quad = F^4 = F^2 = I$$

(3.3) $m^2 = moF^2 + m = m; \mathrm{Im} = mI = 0$

$$FI = IF, Fm - mF = 0$$

These equations shows that there exist two complementary distribution P_1 and $Q_{m;}$ in V corresponding to the projection tensor I and m respectively, when the rank r of F in V, P_1 is r-dimensional and $Q_{m;}$ is $(n-r)$ dimensional, where $\dim.\mathrm{m} = \mathrm{n}$.

We have the following integrability conditions:

(a)A necessary and sufficient condition for the distribution $Q_{m;}$ to be integrble is

(3.4) $\mathrm{N(mX, mY)} = 0$

For any $X, Y \in \Im_1^0$, where N denotes the Nijenhuis tensor of the F-structure.

(b)A necessary and sufficient condition for the distribution P_1 to be integrable is

(3.5) $N(X, Y) = 0$

For any vector field $X, Y \in V$

If the distribution P_1 is integrable , then we have the relation $F^2 = I$, the F operators on V and almost structure F on each integral manifold of D such as $FX' = IX'$, X' being an arbitrary vector field tangent to the integral manifold of P_1, thus rank r of F must be even. When the distribution P_1 is integrable and induced almost complex F is complex analytic on each integral manifold of P_1. Hence we say that F-structure is partially integrable.

(a) A necessary and sufficient condition for the distribution for F-structure to be partially integrable that

(3.6) $N(IX, IY) = 0$ for any $X, Y \in \Im_1^0$.

Let us suppose that there exists, in each coordinates neighbourhoods of V, local coordinates with respect to which F-structure has numerical components

$$F : \begin{bmatrix} 0 & -\mathrm{Im} & 0 \\ \mathrm{Im} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where Im denotes the unit $m \times m$ matrix and $r = 2m$ is the rank of F. If this is the case, we call F-structure F is integrable.

(b)A necessary and sufficient condition for F-structure to be integrable is that

(3.7) $N(X, Y) = 0$

For any vector field X and Y in V.

Let F be a tensor field of type (1,1) in V, then we have, on taking account of (1.21), that the equation $F^3 + F = 0$ is equal to $(F^H)^3 + F^H = 0$,where the rank of F^H is $2r$ and r is the rank of F, thus we have

Theorem 3.1: Let there be given a differentiable manifold, the horizontal lift F^H of an element F of T(V) is an F-structure iff so in F in V, When F is of the rank r in V and F^H is of rank $2r$ inT(V).

Proof: Let there be an F-structure of rank r in V, then the horizontal lift I^H and m^H are two complementary projection in T(V),thus there exists in T(V) two complementary distribution P_1^H and Q_m^H

determined by I^H and m^H respectively .The distribution P_1^H and Q_m^H are respectively lifts of P_1^H and Q_m^H of P_1 and Q_m denoted by N and $\tilde{N}$ the Nijenhuis tensor of F and F^H respectively, then by means of (1.9),(1.18) and (1.24), the conditions (a),(b),(c) and (d)are equivalent to the following conditions:

$(a)'$ $\quad N^H(m^H X^H, m^H Y^H) = 0$

(b)' $\quad m^H N^H(X^H, Y^H) = 0$

(c)' $\quad N^H(I^H X^H, I^H Y^H) = 0$

(d)' $\quad N^H(X^H, Y^H) = 0$

for any vector field X and Y in V, therefore we have,

Theorem 3.2: The Horizontal lift F^H of an F-structure F in V satisfies one of the integrabilty conditions $(a)'$, (b)', (c)' and (d)' in T(V) iff the given F-structure F satisfies the corresponding integrability condition in V.

4. Almost Para contact structure and F-structure: Let there be given n-dimensional differentiable manifold V, a tensor field of type (1,1), a vector field ξ and a co-vector field η satisfying

$$F^2 = I - \xi \otimes \eta;\ F\xi = 0,\ \eta(FX) = 0,\ \eta(\xi) = 1$$

For any vector field X in v, then η is necessarily odd and we call a structure defined by the set (F, ξ, η) of such tensor field F, ξ and η an almost para-contact structure, on taking account of (4.1), we have V= $F^3 + F = 0$ and that F of rank $(n-1)$ everywhere in V. If there is given an almost para-contact structure (F, ξ, η) in V o type (1, 2) is

(4.2) $$S(X,Y) = N(X,Y) + (X(\eta(T)) - Y(\eta(X)) - \eta[X,Y]\xi$$

For any X and Y in V, where N denotes the Nijenhuis tensor of J. The almost para-cotact structure (F, ξ, η) is normal if $S = 0$ .The set is a framed F-structure F in V. If there is given an F-structure F of rank $(n-1)$ in an n-dimensional differentiable manifold V, then there exists an almost Para-contact structure (F, ξ, η) in V. when the framed F-structure (F, ξ) is normal, the given almost Para-contact structure (F, ξ, η) is said to be normal. On taking account of (1.14),(1.16),(1.18) and (1.9), we have from (4.1)

(4.3) $$(F^H)^2 = I - (\xi \otimes \eta^H + \xi^H \otimes \eta^V) = I - \xi^V \otimes \eta^H - \xi^H \otimes \eta^V$$

$$F^V \xi^V = 0; F^H \xi^H = 0; \eta^V (F^H \tilde{X}) = 0$$

$$\eta^H (F^H \tilde{X}) = 0; \eta^V (\xi^H) = 1; \eta^H (\xi^V) = 1; \eta^H (\xi^H) == 0$$

For any vector field $\tilde{X}$ in T(V). Now, if we define tensor field of type (1,1)in T(V) by

(4.4) $$\tilde{J} = F^H + \xi^V \otimes \eta^V - \xi^H \otimes \eta^H$$

Then we obtain $\tilde{J}^2 = -I$ as a direct consequence of (4, 3) ,therefore the tensor field $\tilde{J}$ denoted by (4.) is an almost complex structure in T(V). The almost complex structure $\tilde{J}$ is expressed locally by

$$\tilde{J} = \begin{bmatrix} -\Gamma^h & -\delta^h_\alpha \\ \delta^h_\alpha + \Gamma^\gamma_i \Gamma^h_\gamma & \Gamma^h_\alpha \end{bmatrix}$$

With respect to induced coordinates $(x^h . y^h)$, we have

Theorem4.1: The almost complex structure defined by (4.1) is complex analytic if the almost para-contact structure (F, ξ, η) is normal in V.

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