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Geodetic Dominating Set and Geodetic Domination Polynomials of Extended Grid Graphs

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Abstract : *Let G = (V,E) be a simple graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in $V - S$ is adjacent to atleast one vertex S. Let $D_g(G_n, i)$ be the family of geodetic dominating sets of the graph G_n with cardinality 'i'. Let $dg(G_n, i) = |D_g(G_n, i)|$. In this paper, we obtain a recursive for $d_g(G_n, i)$. Using the recursive formula, we construct the polynomial, $\gamma_g(G_n) = \left\lceil \frac{n+1}{2} \right\rceil$, $\gamma_g(G_n - \{2n\}) = \left\lceil \frac{n+1}{2} \right\rceil$ which we call geodetic dominating polynomial of G_n and obtain some properties of this polynomial.*

Keywords: *Geodetic domination set, geodetic domination number, geodetic domination polynomial*

I . Introduction

Let G = (V, E) be a simple graph of order n. For any vertex $v \in V$, the open neighbourhood of v is the set $N(V) = \{u \in V / uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$. The maximum degree of the graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. A set S of vertices in a graph G is said to be a dominating set if every vertex $v \in V$ is an adjacent to an element of S.The minimum cardinality taken over all geodetic dominating sets S of G is called the geodetic domination number of G and is denoted by $\gamma_g(G)$.A geodetic dominating set with cardinality $\gamma_g(G)$ is called γ_g-set. We denote the set {1, 2,…, 2n–1, 2n} by [2n].

Definition 1 . Consider two paths $[u_1 u_2 \ldots u_n]$ and $[v_1 v_2 \ldots v_n]$. Join each pair of vertices $(u_1 v_1; u_i, v_{i+1}, v_i, u_{i+1})$ i = 1, 2, …, n. The resulting graph is an extended grid graph. The graph given in figure is an Extended grid graph G_n with 2n vertices.

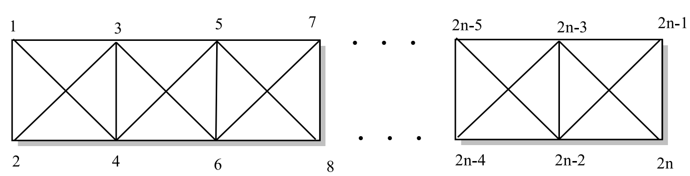

Figure 1.Extended Grid graph G_n

Then, $V(G_n) = \{1, 2, 3, \ldots, 2n-3, 2n-2, 2n\}$ and

$E(G_n)$ = {(1, 3), (3, 5), (5, 7), . . . , (2n – 5, 2n – 3), (2n – 3, 2n – 1),
{2, 4), (4, 6), (6, 8), …, (2n – 4, 2n - 2), (2n – 2, 2n),
(1, 2), (3, 4), (5, 6), …, (2n – 3, 2n – 2), (2n – 1, 2n),
(1, 4), (3, 6), (5, 8), … (2n – 5, 2n – 2), (2n – 3, 2n),
(2, 3), (4, 5), (6, 7), … (2n – 4, 2n – 3), (2n – 2, 2n – 1)}.

For the construction of the geodetic dominating sets of the Extended grid Graphs G_n, we need the geodetic dominating sets of $G_n-\{2n\}$. In this section, we investigate the geodetic dominating sets of G_n. Let $D_g(G_n, i)$ be the family of geodetic dominating of G_n with cardinality i. We shall find the recursive formula for $d_g(G_n, i)$.

Lemma 2.

(i) $\gamma_g(G_n) = \left\lceil \frac{n+1}{2} \right\rceil$,

ii) $\gamma_g(G_n - \{2n\}) = \left\lceil \frac{n+1}{2} \right\rceil$

Lemma 3. For every $n \in N$ and $n \geq 2$ i) $D_g(G_n, i) = \Phi$ if and only if $i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > 2n$

ii) $D_g(G_n- \{2n\}, i) = \Phi$ if and only if $i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > 2n - 1$

Proof . i) It follows from lemma 2(i) and the definition of geodetic dominating set.

ii) It follows from Lemma 2 (ii) and the definition of geodetic dominating sets.

Lemma 4. i) If $D_g(G_n - \{2n\}, i-1) = \Phi$, $D_g(G_{n-1}, i-1) = \Phi$,
$D_g(G_{n-1} - \{2n-2\}, i-1) = \Phi$ and $D_g(G_{n-2}, i-1) \neq \Phi$ then $D_g(G_n, i) \neq \Phi$.

ii) If $D_g(G_n - \{2n\}, i-1) = \Phi$, $D_g(G_{n-1}, i-1) \neq \Phi$;
$D_g(G_{n-2} - \{2n-2\}, i-1) \neq \Phi$; $D_g(G_{n-2}, i-1) \neq \Phi$ then $D_g(G_n, i) \neq \Phi$.

iii) If $D_g(G_n - \{2n\}, i-1) \neq \Phi$, $D_g(G_{n-1}, i-1) \neq \Phi$,
$D_g(G_{n-2} - \{2n-2\}, i-1) \neq \Phi$ and $Dg(G_{n-2}, i-1) \neq \Phi$ then $D_g(G_n, i) \neq \Phi$.

iv) If $D_g(G_n - \{2n\}, i-1) \neq \Phi$, $D_g(G_{n-1}, i-1) = \Phi$,
$D_g(G_{n-1} - \{2n-2\}, i-1) = \Phi$; $D_g(G_{n-2}, i-1) = \Phi$ then $D_g(G_n, i) \neq \Phi$.

v) If $D_g(G_n - \{2n\}, i-1) \neq \Phi$, $D_g(G_{n-1}, i-1) \neq \Phi$,
$D_g(G_{n-1} - \{2n-2\}, i-1) = \Phi$ and $D_g(G_{n-2}, i-1) = \Phi$ then $D_g(G_n, i) \neq \Phi$.

vi) If $D_g(G_n - \{2n\}, i-1) \neq \Phi$, $D_g(G_{n-1}, i-1) \neq \Phi$,
$D_g(G_{n-1} - \{2n-2\}, i-1) \neq \Phi$and $D_g(G_{n-2}, i-1) = \Phi$ then $D_g(G_n, i) \neq \Phi$.

Proof. (i) Since, $D_g(G_n - \{2n\}, i-1) = \Phi$; $D_g(G_{n-1}, i-1) = \Phi$ and $D_g(G_{n-1} - \{2n-2\}, i-1) = \Phi$ by lemma 2 (i) and (ii) we have, $i - 1 < \left\lceil \frac{n+1}{2} \right\rceil$ or $i - 1 > 2n - 1$, $i - 1 < \left\lceil \frac{n}{2} \right\rceil$ or $i - 1 > 2n - 2$, $i - 1 < \left\lceil \frac{n}{2} \right\rceil$ or $i - 1 > 2n - 3$. Therefore , $i - 1 < \left\lceil \frac{n}{2} \right\rceil$ or $i - 1 > 2n - 1$, Therefore, $i < \left\lceil \frac{n+2}{2} \right\rceil$ or $i > 2n$ . . . . . (1).

Also, since $D_g(G_{n-2}, i-1) \neq \phi$, by lemma 3 (i), we have, $\left\lceil \frac{n+1}{2} \right\rceil \leq i \leq 2n - 3$ . . . (2).From (1) and (2), $\left\lceil \frac{n+1}{2} \right\rceil \leq i < \left\lceil \frac{n+2}{2} \right\rceil$. Suppose that $D_g(G_n, i) = \Phi$, then $i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > 2n$ which is a contradiction to the fact that $\left\lceil \frac{n+1}{2} \right\rceil \leq i < \left\lceil \frac{n+2}{2} \right\rceil$. Therefore, $D_g(G_n, i) \neq \Phi$

(ii) Since $D_g(G_n - \{2n\}, i-1) = \Phi$, by lemma 3 (ii) we have $i - 1 < \left\lceil \frac{n+1}{2} \right\rceil$ or $i - 1 > 2n-1$, $i < \left\lceil \frac{n+3}{2} \right\rceil$ or $i > 2n$. . . . . . . (3)

Also, since $D_g(G_{n-1}, i-1) \neq \Phi$, $D_g(G_{n-2} - \{2n-2\}, i-1) \neq \Phi$ and $D_g(G_{n-2}, i-1) \neq \Phi$ by lemma 2 (i) and (ii) we have, $\lceil \frac{n}{2} \rceil \leq i-1 \leq 2n-2$, $\lceil \frac{n-1}{2} \rceil \leq i-1 \leq 2n-3$ and $\lceil \frac{n-1}{2} \rceil \leq i-1 \leq 2n-4$.

Therefore, $\lceil \frac{n}{2} \rceil \leq i-1 \leq 2n-4$ . . . . . . (4)

From (3) and (4), $\lceil \frac{n}{2} \rceil +1 \leq i < \lceil \frac{n+3}{2} \rceil$, $\lceil \frac{n+2}{2} \rceil +1 \leq i < \lceil \frac{n+3}{2} \rceil$, Suppose $D_g(G_n, i) = \Phi$ then by lemma 2.2 (i) we have $i < \lceil \frac{n+1}{2} \rceil$ or $i > 2n$ which is a contradiction to the fact that $\lceil \frac{n+2}{2} \rceil +1 \leq i < \lceil \frac{n+3}{2} \rceil$. Therefore, $D_g(G_n, i) \neq \Phi$

iii) Since, $D_g(G_n - \{2n\}, i-1 \neq \Phi$ $D_g(G_{n-1}, i-1) \neq \Phi$, $D_g(G_{n-1} - \{2n-2\}, i-1) \neq \Phi$ and $D_g(G_{n-2}, i-1) \neq \Phi$ by lemma 2 (i) and (ii) we have, $\lceil \frac{n+1}{2} \rceil \leq i-1 \leq 2n-1$, $\lceil \frac{n}{2} \rceil \leq i-1 \leq 2n-2$, $\lceil \frac{n}{2} \rceil \leq i-1 \leq 2n-3$ and $\lceil \frac{n-1}{2} \rceil \leq i-1 \leq 2n-4$. Therefore, $\lceil \frac{n+1}{2} \rceil \leq i-1 \leq 2n-4$. Hence, $\lceil \frac{n+3}{2} \rceil \leq i \leq 2n-3$. Suppose $D_g(G_n, i) = \Phi$ then by lemma 2 (i), we have $i < \lceil \frac{n+1}{2} \rceil$ or $i > 2n$ which is a contradiction to the fact that $\lceil \frac{n+3}{2} \rceil \leq i \leq 2n-3$. Therefore, $D_g(G_n, i) \neq \Phi$.

iv) $D_g(G_n - \{2n\}, i-1) \neq \Phi$; by lemma 2 (ii) we have $\lceil \frac{n+1}{2} \rceil \leq i-1 \leq 2n-1$. Therefore, $\lceil \frac{n+3}{2} \rceil$
$\leq i \leq 2n$ . . . . . . (5)

Also, since $D_g(G_{n-1}, i-1) = \Phi$; $D_g(G_{n-1} - \{2n-2\}, i-1) = \Phi$ and $D_g(G_{n-2}, i-1) = \Phi$, by lemma 2 (i) and (ii) We have, $i-1 < \lceil \frac{n}{2} \rceil$ or $i-1 > 2n-2$, $i-1 < \lceil \frac{n}{2} \rceil$ or $i-1 > 2n-3$, and $i-1 < \lceil \frac{n-1}{2} \rceil$ or $i-1 > 2n-4$. Therefore, $i-1 < \lceil \frac{n-1}{2} \rceil$ or $i-1 > 2n-2$, $i < \lceil \frac{n+1}{2} \rceil$ or $i > 2n-1$ . . . . (6)

From (5) and (6), $\lceil \frac{n+1}{2} \rceil \leq i < \lceil \frac{n+3}{2} \rceil$. Suppose $D_g(G_n, i) = \Phi$ then by lemma 2 (i), we have $i < \lceil \frac{n+1}{2} \rceil$ or $i > 2n$ which is a contradiction to the fact that $\lceil \frac{n+1}{2} \rceil \leq i < \lceil \frac{n+3}{2} \rceil$. Therefore, $D_g(G_n, i) \neq \Phi$.

v) Since $D_g(G_n - \{2n\}, i-1) \neq \Phi$ and $D_g(G_{n-1}, i-1) \neq \Phi$ by lemma 2 (i) and (ii) we have, $\lceil \frac{n+1}{2} \rceil \leq i-1 \leq 2n-1$ and $\lceil \frac{n}{2} \rceil \leq i-1 \leq 2n-2$. Therefore, $\lceil \frac{n+1}{2} \rceil \leq i-1 \leq 2n-2$. Therefore, $\lceil \frac{n+3}{2} \rceil \leq i \leq$
$2n-1$ . . . . . (7)

Also, since, $D_g(G_{n-1} - \{2n-2\}, i-1) = \Phi$ and $D_g(G_{n-2}, i-1) = \Phi$ by lemma 2 (i) and (ii) we have, $i-1 < \lceil \frac{n}{2} \rceil$ or $i-1 > 2n-3$ and $i-1 < \lceil \frac{n-1}{2} \rceil$ or $i-1 > 2n-4$. Therefore, $i-1 < \lceil \frac{n-1}{2} \rceil$ or $i-1 > 2n-3$, $i < \lceil \frac{n+1}{2} \rceil$ or $i > 2n-2$ . . . . . (8)

From (7) and (8), $2n-3 \leq i < 2n-1$

Suppose $D_g(G_n, i) = \Phi$ then by lemma 2 (i) we have $i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > 2n$ which is a contradiction to the fact that $2n - 3 \le I < 2n - 1$. Therefore, $D_g(G_n, i) \ne \Phi$.

vi) Since, $D_g(G_n - \{2n\}, i - 1) \ne \Phi$; $D_g(G_{n-1}, i - 1) \ne \Phi$ and $D_g(G_{n-1} - \{2n - 2\}, i - 1) \ne \phi$, then by lemma 2 (i) and (ii) we have $\left\lceil \frac{n+1}{2} \right\rceil \le i - 1 \le 2n - 1$, $\left\lceil \frac{n}{2} \right\rceil \le i - 1 \le 2n - 2$ and $\left\lceil \frac{n}{2} \right\rceil \le i - 1 \le 2n - 3$. Therefore, $\left\lceil \frac{n+1}{2} \right\rceil \le i - 1 \le 2n - 3$, $\left\lceil \frac{n+3}{2} \right\rceil \le i \le 2n - 2$ ……..(9)

Also, since, $D_g(G_{n-2}, i - 1) = \Phi$ then by lemma 2 (i) we have, $i - 1 < \left\lceil \frac{n-1}{2} \right\rceil$ or $i - 1 > 2n - 4$

$i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > 2n - 3$ …………..(10)

From (9) and (10), $2n - 3 \le i < 2n - 2$, Suppose $D_g(G_n, i) = \Phi$, then by lemma 2 (i) we have, $i < \left\lceil \frac{n+1}{2} \right\rceil$ or $i > 2n$ which is a contradiction to the fact that $2n\text{-}3 < i \le 2n-1$. Therefore, $D_g(G_n, i) \ne \Phi$

Lemma 5. Suppose that $D_g(G_n, i) \ne \Phi$, then for every $n \in N$,

i) $D_g(G_n - \{2n\}, i - 1) \ne \Phi$, $D_g(G_{n-1}, i - 1) = \Phi$, $D_g(G_{n-1}, -\{2n-2\}, i - 1) = \Phi$, and $D_g(G_{n-2}, i - 1) = \Phi$ if and only if $i = 2n$.

ii) $D_g(G_n - \{2n\}, i - 1) \ne \Phi$, $D_g(G_{n-1}, i - 1) \ne \Phi$, $D_g(G_{n-1}, -\{2n-2\}, i - 1) = \Phi$, and $D_g(G_{n-2}, i - 1) = \Phi$ if and only if $i = 2n - 1$.

iii) $D_g(G_n - \{2n\}, i - 1) \ne \Phi$, $D_g(G_{n-1}, i - 1) \ne \Phi$, $D_g(G_{n-1}, -\{2n-2\}, i - 1) \ne \Phi$, and $D_g(G_{n-2}, i - 1) = \Phi$ if and only if $i = 2n - 2$.

iv) $D_g(G_n - \{2n\}, i - 1) \ne \Phi$, $D_g(G_{n-1}, i - 1) \ne \Phi$, $D_g(G_{n-2}, -\{2n-2\}, i - 1) \ne \Phi$, and $D_g(G_{n-2}, i - 1) \ne \Phi$ if and only if $\left\lceil \frac{n+1}{2} \right\rceil + 1 \le i \le 2n - 3$.

Proof . (i) Since, $D_g(G_{n-1}, i - 1) = \Phi$ and $D_g(G_{n-1} - \{2n - 2\}, i - 1) = \Phi$ by lemma 2 (i) and (ii) we have $i - 1 < \left\lceil \frac{n}{2} \right\rceil$ or $i - 1 > 2n - 2$, $i - 1 < \left\lceil \frac{n}{2} \right\rceil$ or $i - 1 > 2n - 3$, and $i - 1 < \left\lceil \frac{n-1}{2} \right\rceil$ or $i - 1 > 2n - 2$. Suppose, $i - 1 < \left\lceil \frac{n-1}{2} \right\rceil$, then $i < \left\lceil \frac{n+1}{2} \right\rceil$ which implies $D_g(G_n, i) = \phi$ a contradiction. Therefore, $i - 1 > 2n - 2$. Therefore, $i - 1 \ge 2n - 1$ ………. (1)

Also, since, $D_g(G_n - \{2n\}, i - 1) \ne \Phi$, by lemma 2(ii) we have, $\left\lceil \frac{n+1}{2} \right\rceil \le i - 1 \le 2n - 1$ ……… (2)

From (1) and (2), $i - 1 = 2n - 1$ .Therefore, $i = 2n$. Conversely, If $i = 2n$, the $D_g(G_{n-1}, i - 1) = D_g(G_{n-1}, 2n - 1) = \Phi$, $D_g(G_{n-1}, -\{2n - 2\}, i - 1) = D_g(G_{n-1}, -\{2n-2\}, 2n - 1) = \Phi$, $D_g(G_{n-2}, i - 1) = D_g(G_{n-2}, 2n - 1) = \Phi$ and $D_g(G_n - \{2n\}, i - 1) = D_g(G_n - \{2n\}, 2n - 1) \ne \Phi$.

(ii) Assume, $D_g(G_n - \{2n\}, i - 1) \ne \Phi$ and $D_g(G_{n-1}, i - 1) \ne \Phi$, by lemma 2 (i) and (ii) we have, $\left\lceil \frac{n+1}{2} \right\rceil \le i - 1 \le 2n - 1$ and $\left\lceil \frac{n}{2} \right\rceil \le i - 1 \le 2n - 2$ .Therefore, $\left\lceil \frac{n+1}{2} \right\rceil \le i - 1 \le 2n - 2$ ….(3)

Also, since $D_g(G_{n-1}, -\{2n - 2\}, i - 1) = \phi$ and $D_g(G_{n-2}, i - 1) = \phi$, by lemma 2 (i) and (ii) we have, $i - 1 < \left\lceil \frac{n}{2} \right\rceil$ or $i - 1 > 2n - 3$ and $i - 1 < \left\lceil \frac{n-1}{2} \right\rceil$ or $i - 1 > 2n - 4$. Therefore, $i - 1 < \left\lceil \frac{n-1}{2} \right\rceil$ or $i - 1 > 2n - 3$. Suppose, $i - 1 < \left\lceil \frac{n-1}{2} \right\rceil$, then $i - 1 < \left\lceil \frac{n+1}{2} \right\rceil$ which implies, $D_g(G_n, i) = \Phi$ a contradiction. Therefore, $i - 1 > 2n - 3$ .Therefore, $i - 1 \ge 2n - 2$ …..(4)

From (3) and (4) we have $i - 1 = 2n - 2$, Therefore, $i = 2n - 1$, Conversely, if $i = 2n - 1$, then $D_g(G_n - \{2n\}, i - 1) = D_g(G_n - \{2n\}, 2n - 2) \neq \Phi$, $D_g(G_{n-1}, i - 1) = D_g(Gn-1, 2n-2) \neq \Phi$, $_g(G_{n-1} - \{2n-2\}, i - 1) = D_g(G_{n-1} - \{2n - 2\}, 2n - 2) = \Phi$ and $D_g(G_{n-2}, i - 1) = D_g(G_{n-2}, 2n - 2) = \Phi$.

(iii) Since, $D_g(G_n - \{2n\}, i - 1) \neq \Phi$, $D_g(G_{n-1}, i - 1) \neq \Phi$ and $D_g(G_{n-1} - \{2n - 2\}, i - 1) \neq \Phi$, by lemma 2 (i) and (ii) we have $\left\lceil \frac{n+1}{2} \right\rceil \leq i - 1 \leq 2n - 1$, $\left\lceil \frac{n}{2} \right\rceil \leq i - 1 \leq 2n - 2$ and $\left\lceil \frac{n}{2} \right\rceil \leq i - 1 \leq 2n - 3$. Therefore, $\left\lceil \frac{n+1}{2} \right\rceil \leq i - 1 \leq 2n - 3$ …..(5)

Also, since, $D_g(G_{n-2}, i - 1) = \Phi$, by lemma 2 (i) we have $i - 1 < \left\lceil \frac{n-1}{2} \right\rceil$ or $i - 1 > 2n - 4$. Suppose $i - 1 < \left\lceil \frac{n-1}{2} \right\rceil$ then $i < \left\lceil \frac{n+1}{2} \right\rceil$ which implies, $D_g(G_n, i) = \Phi$ a contradiction. Therefore, $i - 1 > 2n - 4$. Therefore, $i - 1 \geq 2n - 3$ …..(6)

From (5) and (6) we have, $i - 1 = 2n - 3$.Therefore, $i = 2n - 2$ .Conversely, if $i = 2n - 2$, then $D_g(G_n - \{2n\}, i - 1) = D_g(G_n - \{2n\}, 2n - 3) \neq \Phi$, $D_g(G_{n-1}, i - 1) = D_g(G_{n-1}, 2n - 3) \neq \Phi$, $D_g(G_{n-1} - \{2n-2\}, i - 1) = D_g(G_{n-1} - \{2n-2\}, 2n - 3) \neq \Phi$ and $D_g(G_{n-2}, i - 1) = D_g(G_{n-2}, 2n - 3) = \Phi$.

iv) Assume $D_g(G_n - \{2n\}, i - 1) \neq \Phi$, $D_g(G_{n-1}, i - 1) \neq \Phi$, $D_g(G_{n-2}, -\{2n-1) \neq \Phi$, and $D_g(G_{n-2}, i - 1) \neq \Phi$. Then by lemma 2 (i) and (ii) we have, $\left\lceil \frac{n+1}{2} \right\rceil \leq i - 1 \leq 2n - 1$, $\left\lceil \frac{n}{2} \right\rceil \leq i - 1 \leq 2n - 2$, $\left\lceil \frac{n}{2} \right\rceil \leq i - 1 \leq 2n - 3$ and $\left\lceil \frac{n}{2} \right\rceil \leq i - 1 \leq 2n - 4$. Therefore, $\left\lceil \frac{n+1}{2} \right\rceil \leq i - 1 \leq 2n - 4$ .Also, since, $D_g(G_n\ i) \neq \Phi$, we have $\left\lceil \frac{n+1}{2} \right\rceil$ or $i - 1 \leq 2n$. But $\left\lceil \frac{n+1}{2} \right\rceil \leq i - 1 \leq 2n - 4$ .Therefore, $\left\lceil \frac{n+1}{2} \right\rceil + 1 \leq i \leq 2n - 4$. Conversely, Suppose $\left\lceil \frac{n+1}{2} \right\rceil + 1 \leq i \leq 2n - 4$. Then by lemma 2 (i) and (ii) we have $\left\lceil \frac{n+1}{2} \right\rceil \leq i - 1 \leq 2n - 1$, $\left\lceil \frac{n}{2} \right\rceil \leq i - 1 \leq 2n - 2$. $\left\lceil \frac{n}{2} \right\rceil \leq i - 1 \leq 2n - 3$ and $\left\lceil \frac{n-1}{2} \right\rceil \leq i - 1 \leq 2n - 4$. From these we obtain that $D_g(G_n - \{2n\}, i - 1) \neq \Phi$, $D_g(G_{n-1}\ i - 1) \neq \Phi$, $D_g(G_{n-2} - \{2n-2\}, i - 1) \neq \Phi$ and $D_g(G_{n-2}, i - 1) \neq \Phi$.

Theorem 6. For every $n \geq 3$ and $i > \left\lceil \frac{n+1}{2} \right\rceil$

i) If $D_g(G_n - \{2n\}, i - 1) \neq \Phi$, $D_g(G_{n-1}, i - 1) = \Phi$, $D_g(G_{n-1} - \{2n - 2\}, i - 1) = \Phi$, $D_g(G_{n-2}, i - 1) = \Phi$then $D_g(G_n, i) = D_g(G_n, 2n) = \{\{1, 2, 3,\ldots, 2n\}\}$.

ii) If $D_g(G_n - \{2n\}, i - 1) \neq \Phi$, $D_g(G_{n-1}, i - 1) \neq \Phi$, $D_g(G_{n-1} - \{2n - 2\}, i - 1) = \Phi$, $D_g(G_{n-2}, i - 1) = \Phi$ then $\{[2n] - \{x\} / x \in [2n]\}$.

iii) If $D_g(G_n - \{2n\}, i - 1) \neq \Phi$, $D_g(G_{n-1}, i - 1) \neq \Phi$; $D_g(G_{n-2} - \{2n - 2\}, i - 1) \neq \Phi$ and $D_g(G_{n-2}, i - 1) \neq \Phi$ then $\{x_1 \cup \{2n\} / x_1 \in D_g(G_n, - \{2n\}, i - 1) \cup x_2 \cup \{2n\} / x_2 \in D_g(G_{n-1}, i - 1) \cup x_3 \cup \{2n\} / x_3 \in D_g(G_{n-1} - \{2n\} \cup x_4 \cup \{2n\} / x_4 \in D_g(G_{n-2}, i - 1)\}$.

Proof. i) Since , $D_g(G_n - \{2n\}, i - 1) \neq \Phi$; $D_g(G_{n-1}, i - 1) = \Phi$, $D_g(G_{n-1} - \{2n-2\}, i - 1) = \Phi$ and $D_g(G_{n-2}, i - 1) = \Phi$ by lemma 5 (i) we have, $i = 2n$. Therefore, $D_g(G_n, i) = D_g(G_n, 2n) = \{\{1, 2, 3, \ldots, 2n\}\}$.

ii) Since , $D_g(G_n - \{2n\}, i - 1) \neq \phi$; $D_g(G_{n-1}, i - 1) \neq \phi$, $D_g(G_{n-1} - \{2n-2\}, i - 1) = \phi$ and $D_g(G_{n-2}, i - 1) = \phi$ then by lemma 5 (ii) we have, $i = 2n - 1$. Therefore, $D_g(G_n, i) = \{[2n] - \{x\} / x \in [2n]\}$

iii) Let $x_1 \in D_g(G_n - \{2n\}, i - 1)$, so atleast one vertex labeled $2n - 1, 2n - 2, 2n - 3$ or $2n - 4$ is in x_1. If $2n - 1, 2n - 2, 2n - 3$ or $2n - 4 \in x_1$, then $x_1 \cup \{2n\} \in D_g(G_n, i)$. Let $x_2 \in D_g(G_{n-1}, i - 1)$, then $2n - 2$, $2n - 3$ or $2n - 4$ is in x_2. If $2n - 2, 2n - 3, 2n - 4 \in x_2$ then $x_2 \cup \{2n - 1\} \in D_g(G_n, i)$. Now let $x_3 \in D_g(G_{n-1} - \{2n-2\}, i - 1)$, then $2n - 3$ or $2n - 4$ is in x_3.If $2n - 3$ or $2n - 4 \in x_3$ then $x_3 \cup \{2n - 2\} \in D_g(G_n, i)$. Now let $x_4 \in D_g(G_{n-2}, i - 1)$, then $2n - 4$ is in x_4. If $2n - 4 \in x_4$ then $x_4 \cup \{2n - 3\} \in D_g(Gn, i)$.

Thus we have $\{x_1 \cup \{2n\} / x_1 \in D_g(G_n - \{2n\} - 1) \cup \{x_2 \cup \{2n\} / x_2 \in D_g(G_{n-1}, i-1)\} \cup \{x_3 \cup \{2n-2\} / x_3 \in D_g(G_{n-1} - \{2n\}, i-1\} \cup \{x_4 \cup \{2n / x_4 \in D_g(G_{n-2}, i-1)\} \subseteq D_g(G_n, i)$ ….. (1)

If $2n \in Y$, then $Y = x_1 \cup \{2n\}$ for some $x_1 \in D_g(G_n - \{2n\}, i-1)$. If $2n \notin Y$ and $2n-1 \in Y$, then $Y = x_2 \cup \{2n\}$ for some $x_2 \in D_g(G_{n-1}, i-1)$. If $2n \notin Y$, $2n-1 \notin Y$ and $2n-2 \in Y$, then $Y = x_3 \cup \{2n\}$ for some $x_3 \in D_g(G_{n-1} - \{2n-2\}, i-1)$. If $2n \notin Y$, $2n-1 \notin Y$ $2n-2 \notin Y$ and $2n-3 \in Y$, then $y = x_4 \cup \{2n\}$ for some $x_4 \in D_g(G_{n-2}, i-1)$.

So $D_g(G_n, i) \subseteq \{x_1 \cup \{2n\} / x_1 \in D_g(G_n - \{2n\}, i-1)\} \cup \{x_2 \cup \{2n\} / x_2 \in D_g(G_{n-1}\, i-1)\} \cup \{x_3 \cup \{2n\} / x_3 \in D_g(G_{n-1} - \{2n-2\}, i-1)\} \cup$ and $\{x_4 \cup \{2n\} / x_4 \in D_g(G_{n-2}, i-1)\}$. …….(2)

From (1) and 2) we have, $D_g(G_n, i) \subseteq \{x_1 \cup \{2n\} / x_1 \in D_g(G_n - \{2n\}, i-1)\} \cup \{x_2 \cup \{2n\} / x_2 \in D_g(G_{n-1}, i-1)\} \cup \{x_3 \cup \{2n\} / x_3 \in D_g(G_{n-1} - \{2n-2\}, i-1)\} \cup \{x_4 \cup \{2n\} / x_4 \in D_g(G_{n-2}, i-1)\}$.

Theorem 7. If $D_g(G_n, i)$ be the family of geodetic dominating sets of G_n with cardinality i , where $i \geq n-2$, then $d_g(G_n,i) = d_g(G_n - \{2n\}, i-1) + d_g(G_{n-1}, i-1) + d_g(G_{n-1} - \{2n-2\}, i-1) + dg(G_{n-2}, i-1)$.

Proof. We consider all the three cases given in theorem 6 By theorem 6 (i),we have, $D_g(G_n,i) = \{x_1 \cup \{2n\} / x_1 \in D_g(G_n - \{2n\}, i-1)\}$. Since , $D_g(G_{n-1},i-1) = \phi$ and $D_g(G_{n-1} - \{2n-2\} = \phi$,We have, , $D_g(G_{n-1},i-1) = 0$ and $D_g(G_{n-1} - \{2n-2\} = 0$.Therefore, $D_g(G_n,i) = d_g(G_n - \{2n\}, i-1)$. By theorem 6(ii) we have, $D_g(G_n,i) = x_1 \cup \{2n-2\} / x_2 \in D_g(G_{n-1}, i-1)\} \cup \{x_3 \cup \{2n-3\} / x_2 \in D_g(G_{n-1} - \{2n-2\}, i-1)\}$. Since , $d_g(G_n,i) = d_g(G_{n-1},i-1) + d_g(G_{n-1} - \{2n-2\}, i-1)$.By theorem 6(iii) we have, $D_g(G_n, i) = \{x_1 \cup \{2n\} / x_1 \in D_g(G_n - \{2n\}, i-1)\} \cup \{x_2 \cup \{2n\} / x_2 \in D_g(G_{n-1}, i-1)\} \cup \{x_3 \cup \{2n\} / x_3 \in D_g(G_{n-1} - \{2n-2\}, i-1)\} \cup \{x_4 \cup \{2n\} / x_4 \in D_g(G_{n-2}, i-1)\}$. Therfore,$d_g(G_n,i) = d_g(G_n - \{2n\}, i-1) + d_g(G_{n-1}, i-1) + d_g(G_{n-1} - \{2n-2\}, i-1) + d_g(G_{n-2}, i-1)$

Definition 8. Let $D_g(G_n, i)$ be the family of geodetic dominating sets of G_n with cardinality i and let $d_g(G_n, i) = |D_g(G_n, i)|$. Then the geodetic domination polynomial $D_g(G_n, x)$ of G_n is defined as, $D_g(G_n, x) =$

$$\sum_{i=\lceil \frac{n+1}{2} \rceil}^{2n} D_g(G_n, i) x^i$$

Theorem 9. For every $n \geq 3$, $D_g(G_n, x) = x\,[D_g(G_n - \{2n\}, x] + D_g(G_{n-1}, x) + D_g(G_{n-1}, - \{2n-2\}, x)]$ with initial values, $D_g(G_2 - \{4\}, = x^3 + x^2$, $D_g(G_2, x) = x^2 + 2x^3 + x^4$,$D_g(G_3 - \{6\}, x) = x^2 + 3x^3 + 3x^4 + x^5$, $D_g(G_3, x) = x^2 + 3x^3 + 6x^4 + 4x^5 + x^6$.

Proof. By theorem 6, we have, $d_g(G_n,i) = d_g(G_n - \{2n\}, i-1) + d_g(G_{n-1}, i-1) + d_g(G_{n-1} - \{2n-2\}, i-1) + d_g(G_n - 2, i-1)$.Therefore, $\Sigma d_g(G_n, i) x^i = \Sigma d_g(G_n, - \{2n\}, i-1) x^i + \Sigma d_g(G_{n-1}, i-1) x^i + \Sigma d_g(G_{n-1} - \{2n-2\}, i-1) x^i + \Sigma d_g(G_{n-2}, i-1) x^i$, $\Sigma d_g(G_n, i) x^i = x \Sigma d_g(G_n, - \{2n\}, i-1) x^{i-1} + x \Sigma d_g(G_{n-1}, i-1) x^{i-1} + x \Sigma d_g(G_{n-1} - \{2n-2\}, i-1) x^{i-1} + x \Sigma d_g(G_{n-2}, i-1) x^{i-1}$. $D_g(G_n, x) = x(D_g(G_n, - \{2n\}, x) + D_g(G_{n-1}, x) + D_g(G_{n-1} - \{2n-2\}, x) + D_g(G_{n-2}, x))$. Therefore, $D_g(G_n, x) = x[D_g(G_n, - \{2n\}, x) + D_g(G_{n-1}, x) + D_g(G_{n-1} - \{2n-2\}, x) + D_g(G_{n-2}, x)]$ With initial values $D_g(G_2 - \{4\}, = x^3 + x^2$, $D_g(G_2, x) = x^2 + 2x^3 + x^4$, $D_g(G_3 - \{6\}, x) = x^2 + 3x^3 + 3x^4 + x^5$, $D_g(G_3, x) = x^2 + 3x^3 + 6x^4 + 4x^5 + x^6$

We obtain $d_g(G_n, i)$ and $d_g(G_n - \{2n\}, i)$ for $2 \leq n \leq 9$ as shown in Table : 1

i / n	2	3	4	5	6	7	8	9	10	11	12		
$G_2 - \{4\}$	1	1											
G_2	1	2	1										
$G_2 - \{6\}$	1	3	3	1									
G_3	1	3	6	4	1								

$G_4-\{8\}$		4	9	10	5	1							
G_4		3	12	19	15	6	1						
$G_5-\{10\}$		2	13	30	34	21	7	1					
G_5		1	12	40	63	55	28	8	1				
$G_6-\{12\}$			10	46	99	117	83	36	9	1			
G_6			6	47	135	211	199	119	45	10	1		
$G_7-\{14\}$			3	41	163	331	404	317	164	55	11	1	
G_7			1	31	174	460	714	714	480	219	66	12	1
$G_8-\{16\}$				20	165	571	1119	1400	1186	698	285	78	13
G8				10	139	637	1573	2269	2550	1875	982	363	91
$G_9-\{18\}$				4	102	641	1999	3810	4700	4216	2847	1344	454
G_9				1	65	580	2809	5465	8193	8916	7008	4180	1797

Table :1

In the following theorem we obtain some properties of $d_g(G_n, i)$.

Theorem 10. The following properties hold for the coefficient of $D_g(G_n, x)$ for all n.

(i) $d_g(G_n, 2n) = 1$ for every $n \geq 2$, ii) $d_g(G_n-\{2n\}, 2n-1) = 1$ for every $n \geq 2$, (iii) $d_g(G_n, 2n-1) = 2n-2$ for every $n \geq 2$, iv) $d_g(G_n, 2n-2) = 2n^2 - 5n + 3$ for every $n \geq 2$.

Proof: (i) As there is only one set of cardinality 2n in G_n ,we have, $d_g(G_n, 2n) = 1$ for every $n \geq 2$

(ii) As there is only one set of cardinality $2n-1$ in $G_n-\{2n\}$,we have $d_g(G_n-\{2n\}, 2n-1) =1$ for every $n \geq 2$.

(iii)To Prove $d_g(G_n, 2n-1) = 2n-2$ for every $n \geq 2$, we apply induction on n .

When n = 2, L.H.S = $d_g(2, 3) = 2$ (from table7.1) and R.H.S = $4(2) - 2 = 2$. Therefore the result is true for n = 2. Now suppose that the result is true for all numbers less than n,by theorem 9,

$$d_g(G_n,2n-1) = d_g(G_n-\{2n-2\},2n-2)+d_g(G_{n-1},2n-2)+d_g(G_{n-1}-\{2n-2\},2n-2) + d_g(G_n-2, 2n-2)$$

$$= 1 + 2(n-2) +1 = 2n-4+2.$$

Therefore, $d_g(G_n,2n-1) = 2n-2$ Hence the result is true for all n.

iv) To Prove $d_g(G_n, 2n-2) = 2n^2 - 5n + 3$ for every $n \geq 2$.we apply induction on 'n' by theorem 9,

$$d_g(G_n, 2n-2) = d_g(G_n-\{2n-2\},2n-3)+ d_g(G_{n-1},2n-3) + d_g(G_{n-1}-\{2n-2\},2n-3) + d_g(G_{n-2}, 2n-3).$$

$$= 2(n-3) +((n-2)^2 - (n-2)+1)+ 2(n-2)+n-1$$

$$= 2n-6+2(n^2-5n+7)+2n-4+ n-1$$

$$= 2n-6+2n^2-10n+7+2n-4+ n-1$$

Therefore, $d_g(G_n, 2n-2) = 2n^2 - 5n + 3$.Hence, the result is true for all n.

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